Dynamic Mode Decomposition

Uri Shaham

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1 Preliminary: similar matrices

Definition 1.1. *Matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$ are called similar if there exist an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$A = P^{-1}BP.$$

Proposition 1.2. *if (λ, u) are an eigenpair of B, then $(\lambda, P^{-1}u)$ are eigenpair of A.*

Proof.

$$AP^{-1}u = P^{-1}BPP^{-1}u = P^{-1}Bu = P^{-1}\lambda u = \lambda(P^{-1}u).$$

□

2 Dynamic Mode Decomposition

DMD is a dimensionality reduction technique for time series, with which we can also analyze the dynamic behavior of the time series and make predictions.

Let $\{v_1, \dots, v_N\}$ be N multivariate (of dimension m observations of a time series, modeled by

$$v_i \approx Av_{i-1},$$

where A is a $m \times m$ matrix. In matrix form, we can write $V_2 = AV_1$, where $V_1 = [v_1, \dots, v_{N-1}]$ and $V_2 = [v_2, \dots, v_N]$. In order to understand the dynamic of the time series and make predictions, we need to estimate A. Let $V_1 = U\Sigma W^T$ be the singular value decomposition of V_1. Then we can write

$$V_2 = AU\Sigma W^T,$$

and multiplying both sides from the left by U^T we have

$$U^T V_2 = U^T AU\Sigma W^T,$$

which we re-arrange to

$$U^T V_2 W\Sigma^{-1} = U^T AU := S.$$

Since A and S are similar, we can compute the eigenvectors and eigenvalues of A from those of S. It is then easy to reconstruct A from its eigendecomposition.

Prediction: once we have A, we can predict v_{N+t} by $A^t v_N$. Also, since $v_i = A^{i-1}v_1 = Q\Lambda^{i-1}Q^T v_1$, where $A = Q\Lambda Q^T$ is the eigendecomposition of A, the series explodes if the largest eigenvalue has magnitude > 1 and vanishes otherwise.

Dimensionality reduction: W can see that the coordinates of v in the basis Q are $\Lambda Q^T v$, i.e., small eigenvalues do not matter much. Hence we can only consider the largest eigenvalues of A.

Analysis: The eigenvalues of A are called modes, and it is common to interpret the eigenvalues as frequencies.

Homework

1. Prove that $A = V_2 V_1^{\dagger}$, where $\dagger$ is the pesudo inverse
2. Create a time series, analyze it using DMD.